Quantum inequalities and 'quantum interest' as eigenvalue problems

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Quantum inequalities (QI's) provide lower bounds on the averaged energy density of a quantum field. We show how the QI's for massless scalar fields in even dimensional Minkowski space may be reformulated in terms of the positivity of a certain self-adjoint operator—a generalized Schrödinger operator with the energy density as the potential—and hence as an eigenvalue problem. We use this idea to verify that the energy density produced by a moving mirror in two dimensions is compatible with the QI's for a large class of mirror trajectories. In addition, we apply this viewpoint to the 'quantum interest conjecture' of Ford and Roman, which asserts that the positive part of an energy density always overcompensates for any negative components. For various simple models in two and four dimensions we obtain the best possible bounds on the 'quantum interest rate' and on the maximum delay between a negative pulse and a compensating positive pulse. Perhaps surprisingly, we find that—in four dimensions—it is impossible for a positive δ-function pulse of any magnitude to compensate for a negative δ-function pulse, no matter how close together they occur.

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I. INTRODUCTION

It is well known that the energy density of a quantum field at a given spacetime point is unbounded from below as a function of the quantum state [1]. However, it turns out that the weighted average of the energy density (either along the worldline of an observer or over a spacetime region) has a lower bound, provided the weight, or *sampling function*, is smooth and has reasonable decay properties. Such bounds are known as *quantum inequalities* (QI's), and place severe constraints on attempts to use quantum fields as a negative energy source in the construction of, for example, wormholes [2] and 'warp drive' spacetimes [3].

In this paper we will consider QI's on the worldline average $\int \rho_\psi(\lambda) f(\lambda) \, d\lambda$, where

$$\rho_\psi(\lambda) := \left\langle T_{ab}^{\text{ren}}(x(\lambda)) \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \right\rangle_\psi \tag{1.1}$$

is the expected energy density in state ψ measured along the worldline $x(\lambda)$ of an observer, and the sampling function f is smooth and positive. The first inequalities of this type were obtained by Ford and Roman [4,5] for fields in Minkowski space in the case where f is the Lorentzian function $L_\tau(t) = \tau/(\pi(t^2 + \tau^2))$. Their results were generalized to static spacetimes by Pfenning and Ford [6]. General sampling functions were first treated by Flanagan [7] for massless field theory in two-dimensional Minkowski space; more recently, Fewster and Eveson [8] used different methods to obtain QI's valid for general sampling functions in Minkowski space of arbitrary dimension and fields of arbitrary mass. This method was extended to static spacetimes by the present authors [9]. Most recently, one of us has proved a rigorous quantum inequality valid for arbitrary smooth worldlines in general globally hyperbolic spacetimes and arbitrary smooth, compactly supported, positive sampling functions [10]. This generalizes and makes precise the results of [8,9]. Quantum inequalities involving spacetime averages have been considered by Helfer [11], who has established their existence under similarly general conditions.

For massless fields in $2m$-dimensional Minkowski space and an observer with worldline $x(t) = (t, \mathbf{0})$, the quantum inequalities of Flanagan [7] and Fewster and Eveson [8] take the form

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$$\int \rho_\psi(t) f(t)\, dt \geq -\frac{1}{c_m} \int |(D^m f^{1/2})(t)|^2\, dt \tag{1.2}$$

where D is the derivative operator and the c_m are constants given by

$$c_m = \begin{cases} 6\pi & m = 1; \\ m\pi^{m-1/2} 2^{2m} \Gamma(m - \frac{1}{2}) & m \geq 2. \end{cases} \tag{1.3}$$

The case $m = 1$ is due to Flanagan and is tighter than the corresponding result of Fewster and Eveson which would give $c_1 = 4\pi$. Note that the right-hand side of Eq. (1.2) is well defined and finite for a large class of sampling functions, including all functions equal to the square of a nonnegative Schwartz test function.

These inequalities imply that ρ_ψ is non-negative on average. For example, putting $f(t) = e^{-\alpha t^2}$ in (1.2) we obtain

$$\liminf_{\alpha \to 0^+} \int \rho_\psi(t) e^{-\alpha t^2}\, dt \geq 0, \tag{1.4}$$

since the right-hand side of (1.2) is of order $\alpha^{m-1/2}$. Ford and Roman [12,13] describe this by way of a financial metaphor: negative portions of $\rho_\psi(t)$ are described as a loan, and positive portions as repayments. The averaged weak energy condition (1.4) becomes the statement that all loans must be repaid in full. However, it is remarkable that the positive contributions actually overcompensate for the negative parts in many examples. The *Quantum Interest Conjecture* of Ford and Roman [12,13] asserts that this is a general phenomenon: quantum 'loans' must be repaid at a positive rate of interest; moreover, there is a maximum allowed delay between the loan and the repayments. In [13] they showed that this is true for the case of a linear combination of two δ-function pulses[1] in two and four dimensions, and also obtained lower bounds on the interest rate and upper bounds on the maximum allowed delay between a negative δ-function pulse and its compensating positive one (the so-called term of the loan). Pretorius [15] has also shown the existence of quantum interest and maximum delays for much more general energy distributions. His work, and that of Ford and Roman, relies on optimizing the appropriate quantum inequality over a one-parameter family of scaled sampling functions, and is not guaranteed to give the best possible bounds.

In this paper we present a new viewpoint on quantum inequalities, which yields the best possible results on quantum interest using the currently available QI's. These results strengthen those of [13,15]. We will study massless fields on $2m$-dimensional Minkowski space, for which the quantum inequality

$$\int \rho_\psi(t) |g(t)|^2\, dt \geq -\frac{1}{c_m} \int |D^m g(t)|^2\, dt \tag{1.5}$$

holds for arbitrary smooth, compactly supported complex-valued functions $g \in C_0^\infty(\mathbb{R})$ and any Hadamard state ψ. For $m \geq 2$, this form of the quantum inequality is a special case of the general result proved in [10] (and also follows from the derivation given in [9] applied to the real and imaginary parts of g separately); in the case $m = 1$ (where the methods of [8–10] give a weaker bound than that of [7]) the inequality is a consequence of Flanagan's result.[2]

Given a candidate energy density ρ, we wish to determine whether or not ρ is the energy density of a physical quantum state. More generally, we may ask whether there exists a state whose energy density agrees with ρ on some (not necessarily bounded) interval $I \subset \mathbb{R}$ of the observer's worldline. The quantum inequality provides an obvious necessary condition: we will say that ρ is *QI-compatible on I* with massless scalar fields in $2m$-dimensions if Eq. (1.5) holds (with ρ_ψ replaced by ρ) for all $g \in C_0^\infty(I)$. Integrating by parts, this condition becomes the inequality

$$\langle g | H^{(m)} g \rangle \geq 0, \qquad \forall g \in C_0^\infty(I) \tag{1.6}$$

where $\langle \cdot | \cdot \rangle$ is the usual L^2-inner product and $H^{(m)}$ is the differential operator

$$H^{(m)} = (-1)^m D^{2m} + c_m \rho. \tag{1.7}$$

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[1] Energy densities of this form are produced in two dimensions by moving mirrors with piecewise constant proper acceleration [13,14].

[2] Suppose $g \in C_0^\infty(\mathbb{R})$. Then $f(t) = |g(t)|^2$ is smooth, compactly supported, and pointwise non-negative, so one may apply Flanagan's bound to f, obtaining a right-hand side equal to $-c_1^{-1} \int (|g|')^2\, dt$. Now $|g|$ is a function of *bounded variation* [16] because $\sum_{k=1}^N ||g(t_k)| - |g(t_{k-1})|| \leq \int |g'(t)|\, dt$ for any finite partition $t_0 < t_1 < \cdots < t_N$ of some closed interval containing the support of g. Accordingly $|g|$ has a finite derivative almost everywhere (see §4 in [16]), and since $(|g|')^2 \leq |g'|^2$ at each such point the inequality (1.5) holds.

At a formal level, then, the QI-compatibility of ρ on I is equivalent to the positivity of $H^{(m)}$ as an operator on $L^2(I)$. Accordingly, the problem is reduced to finding the infimum of the spectrum $\sigma(H^{(m)})$ of $H^{(m)}$, and hence to an eigenvalue problem with ρ as a potential.

Of course, we have skated over various technical issues here, notably the conditions on ρ necessary for $H^{(m)}$ to exist a self-adjoint operator on $L^2(I)$ (especially if ρ is allowed to be distributional[3]), the possibility that inf $\sigma(H^{(m)})$ might be a point of the continuous or singular continuous spectrum, rather than an eigenvalue, and the question of what boundary conditions are appropriate at ∂I (if this is nonempty). These issues are addressed in Sects. II and III, where we describe classes $\mathcal{W}_m$ of candidate energy densities for which the arguments above may be made rigorous. The $\mathcal{W}_m$ are constructed from certain distributional Sobolev spaces, and include progressively more singular distributions as m increases: for example, each $\mathcal{W}_m$ contains the δ-distribution and its first $m-1$ derivatives.

For any $\rho \in \mathcal{W}_m$, we will define $H^{(m)}$ as a self-adjoint operator on $L^2(I)$ by first regarding the expression (1.7) as a sum of quadratic forms. We will then show that the following are equivalent:

1. $\rho \in \mathcal{W}_m$ is QI-compatible on I with massless fields in $2m$-dimensions;

2. The operator $H^{(m)}$ on $L^2(I)$ is positive;

3. $H^{(m)}$ has no strictly negative *eigenvalues*.

The appropriate boundary conditions for the eigenvalue problem require the eigenfunction and its first $m-1$ derivatives to vanish on ∂I.

In fact, the reader who does not wish to be burdened with the details could skip most of Sects. II and III, apart from the definition of $\mathcal{W}_m$ and the statements of Theorems 3.2 and 3.3. However, these sections may be of independent technical interest, as it appears that the quadratic form technique for defining Hamiltonians with singular potentials has previously been applied only to potentials which are locally integrable functions [18,19] (using techniques adapted to this case) or to particular distributions given explicitly as separable interactions (see, e.g., [20], Appendix G of [21] or Example 3 in Section X.2 of [22]). This is largely because the technique has somewhat limited utility for the Schrödinger equation in $d \geq 2$ dimensions, in which case even the δ-function is too singular to be treated.[4] In the present setting, however, the spacetime dimension of the original field theory determines the order of the unperturbed operator $(-1)^m D^{2m}$, and the classes $\mathcal{W}_m$ become more general with increasing m as mentioned above.

As well as reformulating the quantum inequalities, the eigenvalue viewpoint allows us to demonstrate the existence of maximal pulse separations under very general conditions (Theorem 3.3) albeit in a rather nonconstructive fashion. In addition, one may gain more insight into the quantum interest conjecture by analogy with quantum mechanics on the line. Negative energy loans become potential wells; repayments become potential barriers. To be QI-compatible, the barrier must be sufficiently 'large' in some sense to bounce a particle out of the potential well and prevent it being bound. Further comments in this direction are presented in the conclusion.

However, the main thrust of this paper is that our viewpoint provides a practical method for determining QI-compatibility of candidate energy densities, and for obtaining the best possible bounds on maximal pulse separations and quantum interest rates. This is illustrated by various examples, starting in Sect. IV with a simple argument to verify the QI-compatibility of the energy densities arising from a large class of moving mirror trajectories in two dimensions. In Sect. V, we discuss candidate energy densities consisting of isolated δ-function pulses in two dimensions and obtain sharp bounds on the maximal separation between a δ-function loan and repayments (which need not be of δ-function form). For the case where the repayment is also a δ-function pulse, we also determine the minimum quantum interest rate, thereby sharpening the bounds obtained by Ford and Roman [13]. We show that the quantum interest rate becomes unboundedly large as the delay between loan and repayment approaches its maximal (finite) value. By contrast, the lower bound on the interest rate obtained in [13] remains finite in this limit, although this does not contradict our results.

Section VI repeats this analysis in four dimensions. Surprisingly, in this case it turns out that no positive δ-function pulse of any magnitude can compensate for a negative δ-function pulse, no matter how close together they occur. Thus all δ-function combinations of the type considered by Ford and Roman violate the quantum inequality in four dimensions, illustrating the added strength of our method. However, we emphasize that a single negative δ-function pulse can be part of a QI-compatible energy density. To this end, Section VII considers a negative δ-function pulse

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[3]Although the usual (Hadamard) class of physically admissible states yields a smooth energy density, we relax the condition here to allow discussion of, for example, moving mirror models with nonsmooth proper acceleration [14,17].

[4]The δ-distribution is usually treated in terms of self-adjoint extensions (see, e.g. [21]) in $d \leq 3$ dimensions, while more singular distributions may be treated on enlarged, possibly indefinite, inner product spaces (see [23,24] and references therein).

followed at some later time by a constant positive energy density of finite magnitude but infinite duration. Again, it turns out that the magnitude of the compensating pulse must become unboundedly large as the maximal delay is approached.

Our results are all derived for massless fields in even spacetime dimensions. However, as Pretorius [15] has emphasized, the quantum inequalities for massive fields are stronger than those for massless fields (except in two dimensions) and so our results also provide bounds on the massive case. One could interpret the massive QI's directly in terms of a positivity condition, but the analog of H would be a pseudodifferential operator which would be much harder to analyze. Nonetheless, this may prove to be a fruitful viewpoint in deepening our understanding of quantum inequalities.

II. TECHNICAL APPARATUS

In this section we set up the technical framework in which our main results are proved.

A. Sobolev spaces

For completeness and to fix the notation, we briefly describe the various Sobolev spaces which appear in the sequel, mainly following [25].

First, let Ω be any open subset of $\mathbb{R}$, $m = 0, 1, 2, \ldots$ and $1 \le p \le \infty$. The spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are defined as follows: $W^{m,p}(\Omega)$ is the space of functions in $L^p(\Omega)$ whose first m weak derivatives are also in $L^p(\Omega)$, equipped with the norm

$$\|u\|_{m,p} = \left(\sum_{r=0}^{m} \|D^r u\|_p^p \right)^{1/p}, \qquad 1 \le p < \infty,$$
$$\|u\|_{m,\infty} = \max_{0 \le r \le m} \|D^r u\|_\infty, \tag{2.1}$$

while $W_0^{m,p}(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. Here, $\|\cdot\|_p$ denotes the usual L^p-norm on Ω,

$$\|\varphi\|_p = \left(\int_\Omega |\varphi(t)|^p \, dt \right)^{1/p}, \qquad 1 \le p < \infty,$$
$$\|\varphi\|_\infty = \sup_{t \in \Omega} |\varphi(t)|. \tag{2.2}$$

In general $W_0^{m,p}(\Omega)$ is a proper subspace of $W^{m,p}(\Omega)$, but we note the special case $W_0^{m,p}(\mathbb{R}) = W^{m,p}(\mathbb{R})$ $(1 \le p < \infty)$.

An important property of these spaces is given by the Sobolev embedding theorem (Theorem 5.4 in [25]) which we state for the case $p = 2$:

Proposition 2.1. Let I be an open interval of $\mathbb{R}$. For $m \ge 1$, elements of $W^{m,2}(I)$ may be identified with C^{m-1}-functions on the closure of I, and the identification map is continuous. In consequence, elements of $W_0^{m,2}(I)$ vanish along with their first $m-1$ derivatives on the boundary of I.

For $m \ge 0$ and $1 \le p < \infty$, the dual space of $W^{m,p}(\mathbb{R})$ is denoted $W^{-m,p'}(\mathbb{R})$, where the conjugate index p' is defined by

$$p' = \begin{cases} p/(p-1) & 1 < p < \infty; \\ \infty & p = 1. \end{cases} \tag{2.3}$$

This space may also be characterized as the set of distributions in $\mathcal{D}'(\mathbb{R})$ of the form $\omega = \sum_{r=0}^{m} D^r \psi_r$, where each ψ_r belongs to $L^{p'}(\mathbb{R})$. Distributions of this form act antilinearly on $W^{m,p}(\mathbb{R})$ by

$$\omega : f \mapsto \langle f; \omega \rangle = \sum_{r=0}^{m} \int (-1)^r \psi_r(t) \overline{D^r f(t)} \, dt, \tag{2.4}$$

and the norm $\|\cdot\|_{-m,p'}$ is defined to be the operator norm of this map, namely,

$$\|\omega\|_{-m,p'} = \sup_{\substack{f \in W^{m,p}(\mathbb{R}) \\ f \neq 0}} \frac{|\langle f; \omega \rangle|}{\|f\|_{m,p}}. \tag{2.5}$$

Finally, for any $m \in \mathbb{Z}$ the local Sobolev space $W^{m,2}_{\mathrm{loc}}(\Omega)$ is defined to be the space of $u \in \mathcal{D}'(\Omega)$ such that χu is the restriction to Ω of an element of $W^{m,2}(\mathbb{R})$ for all $\chi \in C_0^\infty(\Omega)$. These have the following property:

Proposition 2.2. If f and $D^{2r}f$ both belong to $W^{m,2}_{\mathrm{loc}}(\Omega)$ for some integer $r \geq 1$ then $f \in W^{m+2r,2}_{\mathrm{loc}}(\Omega)$.

See Section IX.6 in [22] (or Theorem 11.1.8 in [26] for a very general setting).

B. Quadratic forms

As mentioned in the introduction, the self-adjoint operator $H^{(m)}$ is defined as a sum of quadratic forms. We briefly review the theory of quadratic forms in Hilbert space [27–29] and then prove two technical results which will underpin our discussion in Sect. III.

Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle \cdot | \cdot \rangle$. A *quadratic form* on $\mathcal{H}$ is a map $a : Q(a) \times Q(a) \to \mathbb{C}$, where $Q(a)$ is a dense linear subset of $\mathcal{H}$ called the *form domain*, with a conjugate linear in the first slot and linear in the second. The form is said to be *semi-bounded* if there exists $M \geq 0$ so that $a(\varphi, \varphi) \geq -M\langle \varphi | \varphi \rangle$ for all $\varphi \in Q(a)$, and *positive* if we may take $M = 0$. A semi-bounded form on a complex Hilbert space is automatically *symmetric*, i.e., $\overline{a(\varphi, \psi)} = a(\psi, \varphi)$ for all $\varphi, \psi \in Q(a)$.

If a is semi-bounded from below by $-M$, it determines an inner product $\langle \cdot | \cdot \rangle_{+1}$ on $Q(a)$ by

$$\langle \varphi | \psi \rangle_{+1} = a(\varphi, \psi) + (M + 1)\langle \varphi | \psi \rangle. \tag{2.6}$$

If $Q(a)$ is complete with respect to the corresponding norm $\| \cdot \|_{+1}$, the form a is said to be *closed* and $Q(a)$ becomes a Hilbert space, sometimes denoted $\mathcal{H}_{+1}$. It is easily seen that $\psi_n \to \psi$ in $\mathcal{H}_{+1}$ if and only if $\psi_n \to \psi$ in $\mathcal{H}$ and $a(\psi_n, \psi_n) \to a(\psi, \psi)$. In consequence, if $\mathcal{D}$ is dense in $Q(a)$ in the $\| \cdot \|_{+1}$ norm, i.e., if $\mathcal{D}$ is a *form core* for a, then a agrees with its continuous extension from $\mathcal{D}$ to $Q(a)$. A particular case of this is:

Proposition 2.3. If a semi-bounded closed form a is positive on a form core $\mathcal{D}$ (i.e., $a(\psi, \psi) \geq 0$ for all $\psi \in \mathcal{D}$) then it is positive.

Proof. For each $\psi \in Q(a)$ there exists a sequence of elements $\psi_n \in \mathcal{D}$ with $\psi_n \to \psi$ in $\| \cdot \|_{+1}$. Thus $a(\psi, \psi) = \lim_{n \to \infty} a(\psi_n, \psi_n) \geq 0$. $\square$

Each semi-bounded form a may be associated uniquely with a self-adjoint operator A whose domain $D(A)$ is contained in $Q(a)$ and such that

$$a(\varphi, \psi) = \langle \varphi | A\psi \rangle \tag{2.7}$$

holds for all $\varphi \in Q(a)$ and $\psi \in D(A)$ (Theorem VI-2.1 of [28]). This operator is specified as follows: if $\varphi \in Q(a)$, $\psi \in \mathcal{H}$, and $a(\chi, \varphi) = \langle \chi | \psi \rangle$ for all χ belonging to a form core for a, then $\varphi \in D(A)$ and $A\varphi = \psi$. In addition, $D(A)$ is itself a form core for a.

On the other hand, if A is a semi-bounded symmetric operator[5] on $\mathcal{H}$, there is a unique closed semi-bounded quadratic form a such that $D(A)$ is a form core for a and Eq. (2.7) holds for all $\varphi, \psi \in D(A)$ (Theorem X.23 in [22]). In turn, a is associated with a self-adjoint operator $\widehat{A}$ called the *Friedrichs extension* of A; it is the unique self-adjoint extension whose domain is contained in $Q(a)$, and the lower bound of its spectrum is equal to the lower bounds of a and A.

Applying this result in the case where A is itself self-adjoint, we establish a bijection between semi-bounded self-adjoint operators and closed semi-bounded quadratic forms (of course, in this case $\widehat{A} = A$). For this reason, the distinction between self-adjoint operators and the corresponding quadratic forms is often blurred, and we speak of the form core of an operator or write $Q(A)$ for the form domain of the quadratic form associated with A.

To illustrate these ideas, let $I \subset \mathbb{R}$ be an open (not necessarily bounded) interval, and consider the operator $(-1)^m D^{2m}$ on $C_0^\infty(I) \subset L^2(I)$, which is easily seen to be symmetric and positive. The form domain of its Friedrichs

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[5]An operator A is *semi-bounded* if there is an $M \geq 0$ such that $\langle \psi | A\psi \rangle \geq -M\|\psi\|^2$ for all $\psi \in D(A)$ and *symmetric* if $\langle \varphi | A\psi \rangle = \langle A\varphi | \psi \rangle$ for all $\varphi, \psi \in D(A)$—note that this is *not* imply that A is self-adjoint (see Ch. VIII in [27]).

extension $A_I^{(m)}$ is therefore the closure of $C_0^\infty(I)$ in the form norm $\|\cdot\|_{+1}^2 = \|D^m \cdot\|_2^2 + \|\cdot\|_2^2$. But this norm is equivalent to the $\|\cdot\|_{m,2}$ Sobolev norm by Corollary 4.16 in [25], so $Q(A_I^{(m)}) = W_0^{m,2}(I)$. Accordingly the elements of $Q(A_I^{(m)})$ are C^{m-1} functions which vanish along with their first $m-1$ derivatives on ∂I; that is, they obey Dirichlet boundary conditions. The domain $D(A_I^{(m)})$ is found as follows: we have $\varphi \in D(A_I^{(m)})$ if and only if $\varphi \in W_0^{m,2}(I)$ and, for some $\psi \in L^2(I)$ we have $\langle\varphi|(-1)^m D^{2m}f\rangle = \langle\psi|f\rangle$ for all $f \in C_0^\infty(I)$. This holds if and only if the $2m$'th weak derivative of φ belongs to $L^2(I)$. We conclude that $D(A_I^{(m)}) = W_0^{m,2}(I) \cap W^{2m,2}(I)$.

We now turn to the definition of operators as sums of quadratic forms. For simplicity, assume A is a positive self-adjoint operator with associated form a. Suppose b is a symmetric quadratic form defined on $Q(A)$. We say that b is *A-form bounded* (or form bounded relative to A) if there exist real constants α, β such that

$$|b(\varphi,\varphi)| \le \alpha a(\varphi,\varphi) + \beta\langle\varphi|\varphi\rangle, \qquad \forall\varphi \in Q(A). \tag{2.8}$$

The *relative form bound* of b with respect to A is the infimum of the set of α for which Eq. (2.8) holds for some choice of β. By the Riesz lemma, if b is A-form bounded then there is a bounded self-adjoint operator $B : \mathcal{H}_{+1} \to \mathcal{H}_{+1}$ such that

$$b(\varphi,\psi) = \langle\varphi|B\psi\rangle_{+1}, \qquad \forall\varphi,\psi \in Q(A). \tag{2.9}$$

If B is compact, then b is said to be *A-form compact*; furthermore, the relative form bound may be shown to vanish in this case (cf. problem 39 in Ch. XIII of [30]). Thus, for each $\alpha > 0$ there exists β such that (2.8) holds.

Our results below will be applications of the following general result. Here, the *essential spectrum* $\sigma_{\text{ess}}(A)$ of a self-adjoint operator A is equal the set of all spectral points of A other than isolated eigenvalues of finite multiplicity.

Theorem 2.4. Let A be a positive self-adjoint operator and let b and c be symmetric quadratic forms defined on $Q(A)$. Suppose that b is positive and A-form bounded, and c is A-form compact. Then $h = a + b + c$ (with form domain $Q(A)$) is the quadratic form of a unique semi-bounded self-adjoint operator H with $Q(H) = Q(A)$. Any form core of A is a form core for H and $\inf\sigma_{\text{ess}}(H) \ge \inf\sigma_{\text{ess}}(A)$. In particular, $\sigma_{\text{ess}}(H) \subset [0,\infty)$.

Proof. By hypothesis on b and c there exist $\alpha, \beta, \gamma > 0$ such that

$$0 \le b(\psi,\psi) \le \alpha a(\psi,\psi) + \beta\|\psi\|^2 \tag{2.10}$$

and

$$|c(\psi,\psi)| \le \frac{1}{2}a(\psi,\psi) + \gamma\|\psi\|^2 \tag{2.11}$$

for all $\psi \in \mathcal{H}_{+1}$. Thus

$$\frac{1}{2}\|\psi\|_{+1}^2 \le \|\psi\|_{+1,h}^2 \le \kappa\|\psi\|_{+1}^2 \tag{2.12}$$

for some $\kappa > 1/2$, where

$$\|\psi\|_{+1,h}^2 = h(\psi,\psi) + (\gamma+1)\|\psi\|^2 \tag{2.13}$$

and we see that $\|\cdot\|_{+1,h}$ and $\|\cdot\|_{+1}$ are equivalent norms. Accordingly, h is a closed semi-bounded quadratic form on $Q(A)$ and is therefore the quadratic form of a unique self-adjoint operator H with $Q(H) = Q(A)$ by Theorem VI-2.1 in [28]. The statement concerning form cores follows because any $\|\cdot\|_{+1}$-dense subset of $Q(A)$ is also $\|\cdot\|_{+1,h}$-dense. Exactly the same argument shows that $h' = a + c$ is the quadratic form of a unique self-adjoint H' with $Q(H') = Q(H) = Q(A)$.

To show that $\inf\sigma_{\text{ess}}(H) \ge \inf\sigma_{\text{ess}}(A)$, we need only to consider the case where H has a nonempty essential spectrum. Since c is A-form compact we have $\sigma_{\text{ess}}(H') = \sigma_{\text{ess}}(A)$ by a generalization of Weyl's theorem[6] so it is enough to show that $\inf\sigma_{\text{ess}}(H) \ge \inf\sigma_{\text{ess}}(H')$. Accordingly, for $n = 1, 2, \ldots$ we define

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[6]See, for example, problem 39 in Ch. XIII of [30]. This may be proved using an argument similar to that of Theorem 6.11 in Ch. 1 of [29].

$$\mu_n = \sup_{\substack{\varphi_1,\ldots,\varphi_{n-1}}} \inf_{\substack{\psi \perp \operatorname{span}\{\varphi_1,\ldots,\varphi_{n-1}\} \\ \|\psi\|=1; \ \psi \in Q(H)}} h(\psi,\psi) \tag{2.14}$$

and μ_n' by the same formula with H and h replaced by H' and h'. Positivity of b (and the fact that $Q(H) = Q(H') = Q(A)$) entails that $\mu_n \geq \mu_n'$ for each n. By the min-max principle (Theorems XIII.1 and XIII.2 in [30]) the μ_n are nondecreasing and tend to $\inf \sigma_{\mathrm{ess}}(H)$ as $n \to \infty$. Hence the μ_n' form a bounded monotonic sequence which therefore converges to $\inf \sigma_{\mathrm{ess}}(H')$, thus yielding the required inequality. $\square$

We remark that the construction of H is based directly on the proof of the KLMN theorem [22] using some ideas drawn from [19].

Corollary 2.5. The following are equivalent: (i) h is positive on a form core for A; (ii) H is positive; (iii) H has no strictly negative eigenvalues.

Proof. Statements (i) and (ii) are equivalent by Proposition 2.3; statements (ii) and (iii) are equivalent because any negative spectral points of H are necessarily isolated eigenvalues since $\sigma_{\mathrm{ess}}(H) \subset [0,\infty)$. $\square$

III. MAIN TECHNICAL RESULTS

We now apply the foregoing results to the theory of quantum inequalities. Let I be any open (not necessarily bounded) interval in $\mathbb{R}$ and, as in Sect. II, let $A_I^{(m)}$ be the Friedrichs extension of the positive symmetric operator $(-1)^m D^{2m}$ on $C_0^\infty(I) \subset L^2(I)$. Each $\rho \in W^{-m,\infty}(\mathbb{R})$ determines a quadratic form ρ_I on $Q(A_I^{(m)})$ by the formula

$$\rho_I(f,g) = \langle (\jmath f)\overline{\jmath g}; \rho \rangle \tag{3.1}$$

and ρ_I is symmetric if ρ is *real* in the sense that $\overline{\langle f; \rho \rangle} = \langle \overline{f}; \rho \rangle$ for all $f \in W^{m,1}(\mathbb{R})$. Here, $\jmath f$ is the element of $W^{m,2}(\mathbb{R})$ agreeing with f on I and vanishing elsewhere.[7]

Comparing with Eq. (1.5), the candidate energy density ρ is QI-compatible on I (with massless scalar fields in $2m$-dimensions) if and only if the form sum $h_I^{(m)} = a_I^{(m)} + c_m \rho_I$ is positive on $C_0^\infty(I)$, i.e., $h_I^{(m)}(g,g) \geq 0$ for all $g \in C_0^\infty(I)$. To turn this into spectral information on a self-adjoint operator we restrict the class of candidate energy densities so as to satisfy the hypotheses of Theorem 2.4.

Accordingly, let $\mathcal{W}_m$ be the set of real $\rho \in W^{-m,\infty}(\mathbb{R})$ such that $\rho = \rho_1 + \rho_2$ with ρ_1 positive (i.e., $\langle f; \rho_1 \rangle \geq 0$ for all pointwise non-negative test functions f) and $\rho_2 \in W^{-m,2}(\mathbb{R}) \cap W^{-m,\infty}(\mathbb{R}) + (W^{-m,\infty}(\mathbb{R}))_\epsilon$.[8] These distributional classes are more than adequate to discuss the examples considered by Ford and Roman [13] and in the present paper. Indeed, $\mathcal{W}_m$ contains the δ-distribution and its first $m-1$ derivatives, and also includes (by virtue of the definition of ρ_2) potentials with very slowly decaying negative tails. The key property of these energy densities is given by the following result, whose proof is given in Appendix A:

Lemma 3.1. Suppose $\rho \in \mathcal{W}_m$ and write $\rho = \rho_1 + \rho_2$ as above. Let $\rho_{i,I}$ be the quadratic forms induced on $Q(A_I^{(m)})$ by the ρ_i according to Eq. (3.1). Then $\rho_{1,I}$ is positive and $A_I^{(m)}$-form bounded and $\rho_{2,I}$ is $A_I^{(m)}$-form compact.

Lemma 3.1 and Theorem 2.4 show that, if $\rho \in \mathcal{W}_m$, then $h_I^{(m)}$ is the quadratic form of a self-adjoint operator $H_I^{(m)}$ with quadratic form domain equal to $Q(A_I^{(m)}) = W_0^{m,2}(I)$ and $\sigma_{\mathrm{ess}}(H_I^{(m)}) \subset [0,\infty)$. We also know that $C_0^\infty(I)$ is a form core for $H_I^{(m)}$. Our main technical result now follows immediately from Corollary 2.5:

Theorem 3.2. Suppose $\rho \in \mathcal{W}_m$ and let I be an open (not necessarily bounded) interval of $\mathbb{R}$. Define $H_I^{(m)}$ as above. Then the following are equivalent: (i) ρ is QI-compatible on I; (ii) $H_I^{(m)}$ is positive; (iii) $H_I^{(m)}$ has no strictly negative eigenvalues.

---

[7]The extension map $\jmath$ is a continuous isometry of $W_0^{m,2}(I)$ into $W^{m,2}(\mathbb{R})$ (Lemma 3.22 in [25]). Formula (3.1) makes sense because the product of two $W^{m,2}$-functions is in $W^{m,1}$ by Leibniz' rule and Hölder's inequality.

[8]That is, for any $\epsilon > 0$, ρ_2 may be written $\rho_2 = \rho_3 + \rho_4$ with $\rho_3 \in W^{-m,2}(\mathbb{R}) \cap W^{-m,\infty}(\mathbb{R})$, and $\rho_4 \in W^{-m,\infty}(\mathbb{R})$ with $\|\rho_4\|_{-m,\infty} < \epsilon$.

The operator domain $D(H_I^{(m)})$ of the operator $H_I^{(m)}$ is the space of $g \in W_0^{m,2}(I)$ for which the distribution $(-1)^m D^{2m} g + \rho g$ may be identified with an element of $L^2(I)$, which we then denote $H_I^{(m)} g$. Here, ρg is to be understood as the distribution acting on $W_0^{m,2}(I)$ by

$$\rho g : f \mapsto \rho_I(f, g). \tag{3.2}$$

It follows that λ is an eigenvalue of $H_I^{(m)}$ if and only if there exists $g \in W_0^{m,2}(I)$ obeying the distributional eigenvalue equation

$$(-1)^m D^{2m} g + c_m \rho g = \lambda g. \tag{3.3}$$

An important part of this result is that it prescribes the regularity and boundary conditions obeyed by g; in particular, by the embedding result Proposition 2.1, g must be at least C^{m-1} and vanish along with its first $m - 1$ derivatives on ∂I. In many cases, however, elliptic regularity arguments allow us to deduce that g enjoys a greater degree of smoothness.

As an example, consider $\rho = \sum_{n=1}^{N} \alpha_n \delta_{\tau_n}$, where δ_{τ_n} is the δ-distribution centered at τ_n, and the α_n are real. Suppose also that the τ_n are contained in the open interval I. For any $m = 1, 2, \ldots$ we have $\rho \in W^{-m,2}(\mathbb{R}) \cap W^{-m,\infty}(\mathbb{R})$ so QI-compatibility reduces to the eigenvalue problem (3.3). Since each $g \in W_0^{m,2}(I)$ is continuous, $\rho g = \sum_{n=1}^{N} \alpha_n g(\tau_n) \delta_{\tau_n}$ and belongs to $W_{\text{loc}}^{-1,2}(I)$. Thus if g solves the eigenvalue problem we have

$$(-1)^m D^{2m} g = \lambda g - \sum_{n=1}^{N} \alpha_n g(\tau_n) \delta_{\tau_n} \in W_{\text{loc}}^{-1,2}(I) \tag{3.4}$$

from which it follows that $g \in W_{\text{loc}}^{2m-1,2}(I)$ by Proposition 2.2 since both g and $D^{2m} g$ belong to $W_{\text{loc}}^{-1,2}(I)$. Thus g has $2m - 2$ continuous derivatives on the closure of I. In general, similar arguments show that g is in fact smooth on any open set excluding the singular support of ρ.

Theorem 3.2 fulfills our stated aim of reducing QI-compatibility to an eigenvalue problem. We will use this viewpoint to obtain quantitative results in the following sections; first, however, we prove the existence of maximal pulse separation times in this setting.

Theorem 3.3. Suppose $\rho \in \mathcal{W}_m$ has compact support $\text{supp} \, \rho$, and let $\mathcal{O}_\rho$ be the set of open intervals I containing $\text{supp} \, \rho$ on which ρ is QI-compatible, ordered by inclusion. If $\mathcal{O}_\rho$ is nonempty, it contains at least one maximal element.

Proof. By Zorn's lemma, it is enough to show that every totally ordered subset of $\mathcal{O}_\rho$ has an upper bound belonging to $\mathcal{O}_\rho$. Let I_α be any totally ordered set of nested open intervals in $\mathcal{O}_\rho$. By Theorem 3.2 each $H_I^{(m)}$ is positive. Because the open interval $I = \cup_\alpha I_\alpha$ is an upper bound for the I_α's it suffices to show that $I \in \mathcal{O}_\rho$. Now, any $f \in C_0^\infty(I)$ belongs to $C_0^\infty(I_\alpha)$ for some (indeed, all sufficiently large) α and so $\langle f | H_I^{(m)} f \rangle = \langle f | H_{I_\alpha}^{(m)} f \rangle \geq 0$. Hence $H_I^{(m)}$ is positive on the form core $C_0^\infty(I)$, and is therefore positive. Thus $I \in \mathcal{O}_\rho$ and the result is proved. $\square$

The significance of this result is that any maximal element of $\mathcal{O}_\rho$ is a maximal interval in which a physical energy density ρ_ψ can agree with ρ. For example, if ρ is supported in $[-t_0, t_0]$, and it turns out that, for some $T > t_0$, $(-\infty, T)$ is a maximal interval in this sense, then any physical energy density ρ_ψ agreeing with ρ on $(-\infty, T)$ must have a compensating pulse starting no later than $t = T$. We note that there may be more than one maximal interval; if, in the above example, ρ was symmetric about the origin, then $(-T, \infty)$ would also be a maximal interval.

To summarize, we have seen that the QI-compatibility of a large class of distributional candidate energy densities on an open interval I may be determined by solving an eigenvalue problem subject to known regularity properties in I and boundary conditions on ∂I. We have also established (in an admittedly nonconstructive fashion) the existence of constraints on 'pulse separation' in this setting. The main importance of these results, to which we now turn, is that they also provide a practical method of verifying QI-compatibility and also obtaining sharp pulse separation bounds and quantum interest rates in particular classes of candidate energy densities.

IV. MOVING MIRRORS IN TWO DIMENSIONS

Suppose an inertial observer in two-dimensional Minkowski space measures the energy density produced by a perfectly reflecting mirror, which is initially at rest in the frame of the observer and subsequently describes a trajectory which is uniform at late times. One would expect the energy density to be QI-compatible (on $\mathbb{R}$) for all 'reasonable'

trajectories—indeed Davies [31] and (in essence) Ford [32] used this assumption as a starting point for the derivation of simple QI bounds; we will now show how our framework allows a straightforward verification of this fact for a large class of trajectories.

We suppose the observer follows a worldline $(t,0)$, and that the mirror follows the worldline $(t, z(t))$ with $z(t) < 0$ for all t to prevent collisions between the observer and mirror. Similar arguments would apply if z was everywhere positive. For simplicity, we assume that z is smooth and that the mirror has boundedly subluminal velocity, i.e., $\sup_t |\dot{z}(t)| < 1$. The assumptions on the initial and final states of motion imply that $z(t) = z_0 < 0$ for all sufficiently large negative t and that $\dot{z}(t)$ vanishes for large positive t. In addition, we must have $\dot{z} \leq 0$ at late times to avoid collisions.[9] As shown by Fulling and Davies [14], the expected energy density in the 'in' vacuum ψ_{in} observed along the worldline $(t,0)$ is given by

$$\rho_{\psi_{\text{in}}}(t) = \frac{1}{12\pi} (p')^{1/2}(t) \left[(p')^{-1/2} \right]''(t), \tag{4.1}$$

where $p(u) = 2\tau_u - u$ and τ_u is defined implicitly by $\tau_u - z(\tau_u) = u$.[10] The function p' is a Doppler shift factor; indeed, writing $\varphi(t) = p'(t)^{-1/2}$, we have

$$\varphi(t) = \sqrt{\frac{1 - z'(\tau_t)}{1 + z'(\tau_t)}}, \tag{4.2}$$

which is smooth, positive and bounded both from above and away from zero. It follows that $\rho_{\psi_{\text{in}}} = (12\pi)^{-1} \varphi''/\varphi$ is smooth and compactly supported (and thus belongs to $\mathcal{W}_1$).

To show that $\rho_{\psi_{\text{in}}}$ is QI-compatible, we must show that the operator $H = -d^2/dt^2 + 6\pi \rho_{\psi_{\text{in}}}$ on $L^2(\mathbb{R})$ is positive. The key observation is that $\rho_{\psi_{\text{in}}}$ may be expressed in terms of the superpotential $U = \varphi'/\varphi$ as

$$\rho_{\psi_{\text{in}}} = \frac{1}{12\pi} \left(U' + U^2 \right). \tag{4.3}$$

Accordingly H may be written in the manifestly positive form

$$H = -\frac{1}{2} \frac{d^2}{dt^2} + A^*A, \tag{4.4}$$

where

$$A = \frac{1}{\sqrt{2}} \left(\frac{d}{dt} - U \right). \tag{4.5}$$

Thus QI-compatibility is (formally) established.

The above argument is easily made rigorous. Since $\rho_{\psi_{\text{in}}}$ is smooth, the definition of H as a sum of forms is equivalent to the ordinary operator sum on $D(H) = W^{2,2}(\mathbb{R})$. The operator A is defined to be the closure of the differential operator (4.5) on $C_0^\infty(\mathbb{R})$, and it is easy to verify that A and A^* act as $\pm D - U$ on their common domain $W^{1,2}(\mathbb{R})$. Straightforward calculation shows that $AD(H) \subset D(A^*)$ and that the identity (4.4) holds on $D(H)$. The operator H is then clearly positive and QI-compatibility follows by Theorem 3.2. Clearly these arguments may also be extended to trajectories which are only C^k for suitable k; we will not pursue this here.

We also note that this argument may be run backwards to show that all energy densities originating from moving mirrors, or from the class of states considered by Ford [32], obey a stronger quantum inequality than that derived by Flanagan [7]; the constant 6π in (1.2) may be replaced by 12π for such states.

V. δ-FUNCTION PULSES IN TWO DIMENSIONS

We now illustrate how the results of Sect. III provide bounds relevant to the quantum interest problem for δ-function loans in two and four dimensions. By solving the appropriate eigenvalue problems we will obtain maximal pulse separations for such a loan, and the minimum quantum interest rates applying to different forms of the repayment.

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[9]These conditions are too restrictive to encompass the nonsmooth trajectory discussed in [13], which also has $\dot{z} > 0$ at large times. The fact that the energy density concerned is QI-compatible (see Sect. V) shows that some of our conditions could be relaxed.

[10]The condition $\sup |\dot{z}| < 1$ guarantees the existence of τ_u.

A. Pulse separation

Consider a loan of the form $\rho(t) = -B\delta(t)$ $(B > 0)$ in two spacetime dimensions $(m = 1)$. Note that $\rho \in \mathcal{W}_1$. This energy density is easily seen not to be QI-compatible on $\mathbb{R}$ (as one would expect) by Theorem 3.2 and the fact that the eigenvalue problem

$$-g'' - 6\pi B\delta(t)g(t) = Eg(t) \tag{5.1}$$

has a solution $g(t) = e^{-3\pi B|t|}$ in $W^{1,2}(\mathbb{R})$ with negative eigenvalue $E = -(3\pi B)^2$. Accordingly, this energy density must be counterbalanced by later repayments or earlier 'deposits' and we may ask how far these positive energy contributions may be separated from it. Theorem 3.3 guarantees the existence of at least one maximal interval containing the origin on which ρ is QI-compatible (provided ρ is QI-compatible on *some* such interval) but does not imply uniqueness. To illustrate this point, we will consider two maximal separation problems. First, we find the maximum T such that ρ is QI-compatible on $(-T, T)$; second, we repeat the analysis for intervals of the form $(-\infty, T)$. The second case is more pertinent to the results of [13], where it was shown that repayments must begin before $T_{\mathrm{FR}} = \pi/(24B) \approx 0.131/B$. As we will see, the eigenvalue approach will provide sharper bounds.

By Theorem 3.2, the solution to our first problem is the maximum T for which the eigenvalue problem (5.1) has no solutions $g \in W_0^{1,2}(-T, T)$ with $E < 0$. Now the discussion in Sect. III implies that any solution to (5.1) must vanish at $t = \pm T$ and be smooth everywhere except at $t = 0$, where it is continuous and satisfies the jump condition $[g']_{t=0} = -6\pi Bg(0)$. These conditions fix g up to normalization as $g(t) = \sinh k(T - |t|)$, and impose the condition

$$kT \coth kT = 3\pi BT \tag{5.2}$$

on $k = \sqrt{-E} > 0$. Since the left-hand side is an increasing function on $\mathbb{R}^+$ and tends to unity as $k \to 0$, we conclude that eigenfunctions with negative eigenvalues exist if and only if $3\pi BT > 1$. Thus the maximal separation here is

$$T_{\mathrm{symm}} = \frac{1}{3\pi B} \approx \frac{0.106}{B}, \tag{5.3}$$

which represents a tighter bound than T_{FR}. The significance of T_{symm} is that if $T > T_{\mathrm{symm}}$ then, no matter what positive energy 'deposits' are made before $t = -T$ or after $t = +T$, the loan of $-B\delta(t)$ at $t = 0$ is forbidden.

In fact this bound may also be obtained [33] by applying the analysis of [13] to the sampling function

$$f_\tau(t) = \begin{cases} \frac{3}{2\tau}(1 - |t|/\tau)^2 & |t| < \tau; \\ 0 & |t| \geq \tau. \end{cases} \tag{5.4}$$

T_{symm} is then the maximum value of τ for which the inequality (1.2) holds with $\rho_\psi(t) = -B\delta(t)$. The function f_τ is a legitimate sampling function because its square root has a weak derivative in L^2 and so belongs to $W^{1,2}(\mathbb{R})$, despite not being differentiable at $t = 0$ and $t = \pm\tau$.

The second problem is to seek the maximum T so that $-B\delta(t)$ is QI-compatible on $(-\infty, T)$. Solving the eigenvalue problem (5.1) for $g \in W_0^{1,2}(-\infty, T)$ we find

$$g(t) = \begin{cases} e^{kt} \sinh kT & t < 0; \\ \sinh k(T - t) & 0 < t < T \end{cases} \tag{5.5}$$

up to normalization, where $k = \sqrt{-E} > 0$ as before. Here we have used the fact that g must be square integrable and also continuous at $t = 0$. Applying the jump condition gives

$$6\pi BT = kT(1 + \coth kT), \tag{5.6}$$

which fails to have solutions with $k > 0$ if and only if

$$T \leq T_{-\infty} := \frac{1}{6\pi B} \approx \frac{0.0531}{B}. \tag{5.7}$$

This is the quantity most relevant to the problem studied by Ford and Roman [13]: if the energy density vanishes for all $t < 0$ and a negative δ-loan is made at $t = 0$, repayments must begin before $t = T_{-\infty}$.

In the terms of Theorem 3.3, both $(-T_{\mathrm{symm}}, T_{\mathrm{symm}})$ and $(-\infty, T_{-\infty})$ are maximal elements of $\mathcal{O}_\rho$. The fact that $T_{-\infty} < T_{\mathrm{symm}}$ is of course necessary for consistency.

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B. Quantum interest

Let us now consider what happens if a δ-function compensating pulse arrives at time $T \leq T_{-\infty}$. For what values of the 'interest rate' ϵ is the combined energy density $B[-\delta(t) + (1+\epsilon)\delta(t-T)]$ QI-compatible on $\mathbb{R}$?

By Theorem 3.2 we require there to be no solutions in $g \in W_0^{1,2}(\mathbb{R})$ to

$$-g''(t) + [-\lambda\delta(t) + \mu\delta(t-T)]g(t) = -k^2 g(t) \tag{5.8}$$

where $\lambda = 6\pi B$, $\mu = 6\pi(1+\epsilon)B$ and we take $k > 0$ without loss of generality. Any solution would be square integrable and smooth except at $t = 0, T$ where it would be continuous and obey the jump conditions $[g']_{t=0} = -\lambda g(0)$ and $[g']_{t=T} = \mu g(T)$. These requirements fix g up to normalization as

$$g(t) = \begin{cases} e^{kt} & t \leq 0; \\ \cosh kt + (1 - \lambda/k)\sinh kt & 0 < t \leq T; \\ [\cosh kT + (1 - \lambda/k)\sinh kT]\, e^{-k(t-T)} & T < t, \end{cases} \tag{5.9}$$

and impose the condition $f_{\lambda T, \mu T}(kT) = 0$ on k, where

$$f_{\beta,\gamma}(x) = (2x - \beta + \gamma)\cosh x + \left(2x + \gamma - \beta - \frac{\beta\gamma}{x}\right)\sinh x. \tag{5.10}$$

Accordingly, ρ is QI-compatible on $\mathbb{R}$ if and only if $f_{\beta,\gamma}(x)$ is nonzero for all real $x > 0$, which is equivalent to the conditions

$$0 < \beta < 1 \qquad \text{and} \qquad \gamma \geq \beta/(1-\beta), \tag{5.11}$$

as we now show. Sufficiency holds because each coefficient in the power series

$$f_{\beta,\gamma}(x) = \gamma - \beta - \beta\gamma + \sum_{n=1}^{\infty} x^{2n} \left(\frac{\gamma - \beta}{(2n)!} - \frac{\beta\gamma}{(2n+1)!} + \frac{2}{(2n-1)!} \right)$$
$$+ \sum_{n=0}^{\infty} x^{2n+1} \left(\frac{2}{(2n)!} + \frac{\gamma - \beta}{(2n+1)!} \right) \tag{5.12}$$

is positive under these conditions. On the other hand, if $f_{\beta,\gamma}$ is nonvanishing for $x > 0$, then the intermediate value theorem implies $f_{\beta,\gamma}(0) \geq 0$ since $f_{\beta,\gamma}$ is everywhere smooth and is positive for large positive x. Thus $\gamma(1-\beta) \geq \beta$, and the required result follows as we also have $0 < \beta \leq 1$ from the condition $0 < T \leq T_{-\infty}$.

In terms of the original parameters, condition (5.11) reads

$$0 < BT < \frac{1}{6\pi} \qquad \text{and} \qquad \epsilon \geq \frac{6\pi BT}{1 - 6\pi BT}. \tag{5.13}$$

Note that the minimum interest rate $6\pi BT/(1 - 6\pi BT)$ grows unboundedly as the duration of the loan approaches its maximum. This is a stronger result than that of Ford and Roman [13], whose lower bound on the interest rate remains finite as the duration of the loan approaches T_{FR}.

Finally, we note that the particular moving mirror trajectory considered in [13] has (in our notation) $BT < 1/(12\pi)$ and

$$\epsilon = \frac{24\pi BT[72\pi^2(BT)^2 - 15\pi BT + 1]}{(1 - 12\pi BT)^3}, \tag{5.14}$$

which easily satisfies the conditions above. The corresponding energy density is therefore QI-compatible on $\mathbb{R}$, in accord with the results of Sect. IV (although the trajectory itself does not satisfy the conditions imposed there). This example also exhibits divergent quantum interest rates, but the divergence here occurs before the duration reaches $T_{-\infty}$.

VI. δ-FUNCTION PULSES IN FOUR DIMENSIONS

A. Pulse separation

We compute maximum pulse separations for the distribution $\rho = -B\delta(t)$ in four-dimensional massless field theory ($\rho \in \mathcal{W}_2$). First, the maximal symmetric interval $(-T, T)$ on which ρ is QI-compatible is found as follows. The appropriate eigenvalue problem is

$$g''''(t) - 16\pi^2 B\delta(t)g(t) = -4k^4 g(t) \tag{6.1}$$

which is to be solved for $k > 0$ and $g \in W_0^{2,2}(-T, T)$. Both g and g' vanish at the endpoints, and g is smooth except at $t = 0$, where it is C^2 with a discontinuity in g''' determined by the coefficient of the δ-function. The boundary conditions and continuity of g, g' and g'' at $t = 0$ fix g up to normalization as

$$g(t) = 2\Psi(kT)\Psi\left(k(T - |t|)\right) + \Phi^+(kT)\Phi^-\left(k(T - |t|)\right) \tag{6.2}$$

where

$$\begin{aligned}
\Psi(x) &= \sinh x \sin x, \\
\Phi^\pm(x) &= \sinh x \cos x \pm \cosh x \sin x,
\end{aligned} \tag{6.3}$$

so the discontinuity in g''' at $t = 0$ is

$$[g''']_{t=0} = 4k^3(\sin 2kT + \sinh 2kT). \tag{6.4}$$

Accordingly, the eigenvalue condition $[g''']_{t=0} = 16\pi^2 B g(0)$ is

$$f(2kT) = 16\pi^2 BT^3 \tag{6.5}$$

where

$$f(x) = x^3 \frac{\sinh x + \sin x}{\cosh x + \cos x - 2}. \tag{6.6}$$

Now f is increasing on $(0, \infty)$ and tends to 24 as $x \to 0^+$, so the eigenvalue equation fails to have solutions if and only if $16\pi^2 BT^3 \leq 24$. Thus $T_{\text{symm}} = (3/(2\pi^2 B))^{1/3}$.

Repeating the analysis for QI-compatibility on $(-\infty, T)$, we find eigenfunctions as follows:

$$g(t) = \begin{cases} 2\sin kT\,\Psi(k(t - T)) - (\sin kT + \cos kT)\Phi^-(k(t - T)) & t > 0; \\ e^{k(t-T)}(C \sin kt - D \cos kt) & t < 0, \end{cases} \tag{6.7}$$

where $C = \frac{1}{2}(\sin 2kT + \cos 2kT - e^{2kT})$, $D = \frac{1}{2}(\sin 2kT - \cos 2kT - e^{2kT} + 2)$. The jump condition becomes $f(2kT) = 16\pi^2 BT^3$ where

$$f(x) = \frac{x^3}{1 + e^{-x}(\cos x - \sin x - 2)}. \tag{6.8}$$

This function obeys $f(x) \geq 3$ for $x \geq 0$ with equality at $x = 0$. Thus the maximal pulse separation here is

$$T_\infty = \left(\frac{3}{16\pi^2 B}\right)^{1/3} \approx 0.267 B^{-1/3} \tag{6.9}$$

which improves on Ford and Roman's upper bound $0.338 B^{-1/3}$ [13]. We emphasize that this bound strengthens (and does not contradict) their result.[11] Our assertions concerning f are proved by the following means: let $q(x)$ be the denominator, then $q(0) = q'(0) = q''(0) = 0$ and $q'''(x) = (2 - 4\sin x)e^{-x} \leq 2$ for all $x \in \mathbb{R}^+$, so Taylor's theorem with remainder shows

$$q(x) \leq \frac{x^3}{3} \tag{6.10}$$

on $\mathbb{R}^+$. Since $q \geq 0$ on $\mathbb{R}^+$, we have $3 \leq f(x)$ with equality in the limit $x \to 0$.

---

[11]Incidentally, it turns out that the square root of the sampling function employed in [13] is not an element of $W^{2,2}(\mathbb{R})$ and would not be a legitimate sampling function in our framework. Nonetheless, it is certainly true that their bound is a valid upper bound for the pulse separation. This issue does not arise in two dimensions because the square root of the sampling function of [13] is an element of $W^{1,2}(\mathbb{R})$.

B. Quantum interest

Repeating the analysis of Sect. VB, we consider the QI-compatibility of $\rho(t) = B[-\delta(t) + (1+\epsilon)\delta(t-T)]$ by seeking solutions $g \in W_0^{2,2}(\mathbb{R})$ to the equation

$$g''''(t) + [-\lambda\delta(t) + \mu\delta(t-T)]g(t) = -4k^4 g(t) \tag{6.11}$$

where $\lambda = 16\pi^2 B$, $\mu = 16\pi^2 B(1+\epsilon)$, and $k > 0$. On $(-\infty, T)$, any solution must take the form

$$g(t) = e^{kt}(C\cos kt + D\sin kt) - C\theta(t)\frac{\lambda}{4k^3}\Phi^-(kt) \tag{6.12}$$

while on (T, ∞) we have

$$g(t) = e^{-kt}(E\cos kt + F\sin kt). \tag{6.13}$$

The matching conditions at $t = T$—continuity of g, g', and g'', and a discontinuity $[g''']_{t=T} = -\mu g(T)$ in g'''—fix C, D, E, F up to overall scale as

$$
\begin{aligned}
C &= 8k^3(\cos kT + \sin kT), \\
D &= (8k^3 - 2\lambda)\sin kT - 8k^3\cos kT, \\
E &= [(8k^3 - \lambda)e^{2kT} + \lambda]\cos kT + [(\lambda - 8k^3)e^{2kT} + \lambda]\sin kT, \\
F &= [(8k^3 - \lambda)e^{2kT} + \lambda](\sin kT + \cos kT),
\end{aligned} \tag{6.14}
$$

and impose the condition $f_{\lambda T^3, \mu T^3}(2kT) = 0$ on k, where

$$f_{\beta,\gamma}(x) = x^6 + x^3(\gamma - \beta) + \beta\gamma[e^{-x}(1 + \sin x) - 1] = 0. \tag{6.15}$$

Since $f_{\beta,\gamma}(x) \to +\infty$ as $x \to +\infty$ and $f_{\beta,\gamma}(x) = -\frac{1}{2}\beta\gamma x^2 + O(x^3)$ for small x, we see that this condition has a solution for any positive values of λ and μ.

We conclude that for any $B > 0$ there is no 'interest rate' ϵ for which the energy density considered here can be QI-compatible in four dimensions, and that there is no physical state with an expected energy density of this form. (Note that one would not expect to produce such a simple energy density by moving mirrors in four dimensions; other contributions also appear, at least in the nonrelativistic approximation [17].)

A legitimate question at this stage is: what does this result tell us about more physical smooth candidate energy densities?[12] In other words, can one also rule out energy densities consisting of two strongly peaked, but nonetheless smooth, pulses? We hope to address this in detail elsewhere and give only a qualitative answer here. Suppose one is given a sequence ρ_n of smooth, compactly supported functions which converge to $\rho(t) = B[-\delta(t) + (1+\epsilon)\delta(t-T)]$ for some values of B, T, ϵ in the $\|\cdot\|_{-2,\infty}$-norm.[13] Then Theorem B.1 of Appendix B shows that the corresponding operators $H_n^{(2)} = D^4 + c_2\rho_n$ on $L^2(\mathbb{R})$ converge to the limiting operator $H^{(2)} = D^4 + c_2\rho$ in the *norm resolvent sense* (see §VIII.7 in [27]). Now the spectrum of self-adjoint operators cannot expand in a norm resolvent limit (indeed, the same is true for the weaker strong resolvent convergence—see Theorem VIII.24 in [27]) so, since $H^{(2)}$ has at least one negative eigenvalue, so must $H_n^{(2)}$ for all sufficiently large n. Accordingly, the ρ_n fail to be QI-compatible for all sufficiently large n.

Thus we see that our result does have implications for smooth candidate energy densities, although the precise quantitative details require further work.

VII. δ-FUNCTION PULSE WITH A POSITIVE STEP IN FOUR DIMENSIONS

Our last example concerns a negative-energy δ-function pulse at $t = 0$ followed at time $t = T$ by a positive energy density of finite magnitude but infinite duration

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[12] We are grateful to the referee for raising this issue.

[13] Such sequences are easily obtained by mollifying ρ, cf. Theorem 4.1.4 in [34] and Lemma 3.15 in [25].

$$\rho(t) = -B\left[\delta(t) + \frac{1+\epsilon}{\tau}\theta(t-T)\right] \tag{7.1}$$

for $B, \epsilon > 0$, where τ is a timescale inserted for dimensional reasons. This candidate energy density belongs to $\mathcal{W}_2$ because the second term is a positive element of $L^\infty(\mathbb{R})$ and hence of $W^{-2,\infty}(\mathbb{R})$. Note that the total 'repayment' of positive energy is infinite for all values of these parameters; nonetheless we will see that QI-compatibility forces ϵ to increase without bound as T approaches the maximal value $T_{-\infty}$ obtained above.

It will be convenient to introduce dimensionless quantities $\beta = 16\pi^2 BT^3$ and $\gamma = [64\pi^2 B(1+\epsilon)\tau^{-1}T^4]^{1/4}$. For $t < T$ the analysis is identical to that in the preceding section and the eigenfunction g takes the form (6.12) in this range. When $T < t$, the eigenvalue problem is

$$g'''' = -4k'^4 g, \tag{7.2}$$

where $k'^4 = k^4 + (\gamma/2T)^4$ so the solution takes the general form

$$g(t) = e^{-k't}[E\cos k'(t-T) + F\sin k'(t-T)], \tag{7.3}$$

since g must be square integrable. Continuity of g, g', g'' at $t = T$ fixes the coefficients up to scale as

$$C = x^3(x+y)\left(x\cos\frac{x}{2} + y\sin\frac{x}{2}\right)e^{x/2},$$

$$D = 2\beta\left[y^2\Phi^-(x/2) - x^2\Phi^+(x/2)\right] - x^3(x+y)\left(y\cos\frac{x}{2} - x\sin\frac{x}{2}\right)e^{x/2},$$

$$E = \frac{Ce^{y/2}}{x^3}\left[x^3e^{x/2}\cos\frac{x}{2} - 2\beta\Phi^-(x/2)\right] + De^{(x+y)/2}\sin\frac{x}{2},$$

$$F = \frac{Ce^{y/2}}{xy^2}\left[x^3e^{x/2}\sin\frac{x}{2} - 2\beta\Phi^+(x/2)\right] - D\frac{x^2}{y^2}e^{(x+y)/2}\sin\frac{x}{2}, \tag{7.4}$$

where $x = 2kT$ and $y = 2k'T = (x^4 + \gamma^4)^{1/4}$. The final matching requirement, continuity of g''' at $t = T$, imposes the eigenvalue condition $f_{\beta,\gamma}(x) = 0$ on x where $f_{\beta,\gamma}$, defined by

$$f_{\beta,\gamma}(2kT) = \frac{2T^2(k'-k)}{k(k+k')}[g''']_{t=T} \tag{7.5}$$

may be shown, after a tedious calculation, to take the form

$$f_{\beta,\gamma}(x) = \gamma^4\left[x^3e^x + \beta\left(2 - \cos x + \sin x - e^x\right)\right]$$
$$+4\beta x(y-x)\left[y^2(\cos x - 1) - xy\sin x - x^2\right]. \tag{7.6}$$

The QI-compatibility of ρ is equivalent to the absence of zeros of $f_{\beta,\gamma}(x)$ for $x > 0$. Since $f_{\beta,\gamma}(x) \to +\infty$ as $x \to +\infty$, a necessary condition for QI-compatibility is that $f_{\beta,\gamma}(x)$ should be non-negative for small $x > 0$. Now

$$f_{\beta,\gamma}(x) = \gamma\left[\gamma^3 - \beta\left(4 + 4\gamma + 2\gamma^2 + \frac{1}{3}\gamma^3\right)\right]x^3 + O(x^4), \tag{7.7}$$

so our necessary condition becomes

$$P(\gamma) := \gamma^3 - \beta\left(4 + 4\gamma + 2\gamma^2 + \frac{1}{3}\gamma^3\right) \geq 0. \tag{7.8}$$

In addition, the pulse separation bound of Sect. VI A entails that $0 < \beta \leq 3$. It is now easy to see that condition (7.8) implies

$$0 < \beta < 3 \qquad \text{and} \qquad \gamma \geq \gamma_0 \tag{7.9}$$

as necessary conditions for QI-compatibility. Here, γ_0 is the unique real root of $P(\gamma)$, given by

$$\gamma_0 = \frac{2\beta}{3-\beta} + \frac{4\sqrt{3\beta}}{3-\beta}\cosh\left[\frac{1}{3}\cosh^{-1}\left(\frac{\beta^2+27}{12\sqrt{3\beta}}\right)\right]. \tag{7.10}$$

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Note that γ_0 increases monotonically and without bound as the maximal loan term $\beta = 3$ is approached.

In fact, the conditions (7.9) are not only necessary, but also sufficient for QI-compatibility as we now show. To do this, we must prove that $f_{\beta,\gamma}(x)$ does not vanish for $x > 0$ if (7.9) holds. Now, the analysis at the end of Sect. VI A concerning the function f implies that

$$2 - \cos x + \sin x - e^x \geq -\frac{x^3}{3}e^x. \tag{7.11}$$

Using this inequality, along with condition (7.8) in the form

$$\gamma^4 \geq \beta\gamma\left(4 + 4\gamma + 2\gamma^2 + \frac{1}{3}\gamma^3\right), \tag{7.12}$$

we have

$$f_{\beta,\gamma}(x) \geq 2\beta\gamma(\gamma^2 + 2\gamma + 2)x^3 e^x + 4\beta x(y - x)\left[y^2(\cos x - 1) - xy\sin x - x^2\right]$$
$$= \kappa\left[\left(\frac{1}{2}\gamma^2 + \gamma + 1\right)(y + x)(y^2 + x^2)e^x + \gamma^3\left(y^2\frac{\cos x - 1}{x^2} - y\frac{\sin x}{x} - 1\right)\right]$$
$$\tag{7.13}$$

where

$$\kappa = \frac{4\beta\gamma x^3}{(y + x)(y^2 + x^2)} \tag{7.14}$$

is positive.

Next, applying the elementary bounds $(\sin x)/x \leq 1$ and $(\cos x - 1)/x^2 \geq -1/2$ for $x > 0$, we obtain

$$f_{\beta,\gamma}(x) \geq \kappa'\left[(y + x)\left(1 + \frac{x^2}{y^2}\right)e^x - \frac{\gamma}{\frac{1}{2}\gamma^2 + \gamma + 1}\left(\frac{\gamma^2}{2} + \frac{\gamma^2}{y} + \frac{\gamma^2}{y^2}\right)\right] \tag{7.15}$$

where

$$\kappa' = \kappa y^2\left(\frac{1}{2}\gamma^2 + \gamma + 1\right) > 0. \tag{7.16}$$

Finally, we have $y > \gamma$, so the first term in the braces is greater than γ [because, additionally, $x > 0$ and $e^x > 1$] while the second term is less than γ. Hence $f_{\beta,\gamma}(x) \geq 0$ for all $x > 0$ as required.

To summarize, we restate the QI-compatibility conditions (7.9) in terms of the original parameters B, T, and ϵ, in which form they read

$$0 < BT^3 < \frac{3}{16\pi^2} \qquad \text{and} \qquad B(1 + \epsilon) \geq \frac{\gamma_0^4\tau}{64\pi^2 T^4}. \tag{7.17}$$

With B fixed, it is clear that $\epsilon \to \infty$ as T approaches the maximal value $T_{-\infty}$, because $\gamma_0 \to \infty$ as $\beta \to 3$.

VIII. CONCLUSION

We have presented a new viewpoint on quantum inequalities, in terms of eigenvalue problems, and have demonstrated its utility with reference to the circle of ideas surrounding the quantum interest conjecture. In particular, we have obtained optimal bounds on the pulse separations and quantum interest rates which improve on the values found in [13] and indeed rule out the double δ-function pulse model in four dimensions. Our general approach has also given a simple confirmation that moving mirrors in two dimensions yield QI-compatible energy densities for a large class of trajectories. We conclude with a few remarks.

First, we have seen that quite singular candidate energy densities, in particular (derivatives of) δ-functions, may be accommodated within our framework. Although the classes $\mathcal{W}_m$ studied in Sect. III are more general than strictly required to treat the examples studied here, they strongly suggest the possibility of extending the results of [10] to (certain classes of) non-Hadamard states.

Second, it is noteworthy that sampling functions, which were the key element allowing Ford and coworkers to progress beyond the pointwise unboundedness of the energy density, are almost eliminated from the present treatment. The eigenvalue viewpoint may thus be considered as a 'coordinate-free' version of the quantum inequalities. Of course, there will be many circumstances in which the eigenvalue problem cannot be solved analytically. One would then need to fall back on the use of sampling functions much as in [13,15]. However, the eigenvalue viewpoint can still be of use, as the strongest information regarding QI-compatibility will be obtained from sampling functions of the form $|g(t)|^2$, where g is chosen to be a good approximation to an eigenfunction e.g., by WKB methods.

Third, our viewpoint provides a more intuitive understanding of the quantum interest conjecture using the analogy with quantum mechanics for a particle moving on a line. This is a precise correspondence for two-dimensional quantum fields, but much of the intuition is also valid in the four-dimensional case. A negative energy loan may be considered as a potential well in which bound states would form in the absence of a suitable positive energy repayment, which acts as a barrier. To repay the loan, the barrier must disrupt the tail of the bound state wavefunction, essentially bouncing the particle out of the potential well. Since the tail decays rapidly away from the well, it is not surprising that the repayment must be greater than the original loan in some sense, and that it must become larger as the term of the loan is increased. In a precise sense *no* positive barrier can have a greater effect than a perfectly reflecting wall. The maximum pulse separation is therefore the minimum delay between the well and a wall such that bound states may still form. In the examples studied here, the interest rate diverges as the maximum loan term is approached. We believe this to be very natural: given the existence of maximal pulse separations there must be a physical mechanism which prevents one from exceeding this bound, and it seems more natural that the interest rate should diverge as one approaches the limit rather than for such a mechanism to switch on suddenly at the maximal separation. We intend to return to this issue elsewhere.

To give a direct physical interpretation of our results in terms of quantum field theory, consider the problem of verifying that a given candidate energy density is the energy density of a quantum state. It would be necessary to construct a two-point function with normal ordered energy density agreeing with the candidate on a specified worldline. In effect, one would be attempting to solve the Klein-Gordon equation subject to data specified on this worldline, for a bisolution obeying the positivity conditions necessary for it to arise from a state (and which, in turn, imply the quantum inequalities [10]). This is a somewhat involved process and it is not easy to gain direct insight into the class of functions for which it is possible. The results obtained here and in [13,15] suggest very strongly that these conditions are highly restrictive and also function in a semi-local fashion: if the bisolution develops a region in which it fails to obey positivity, this cannot be repaired by modifying the candidate energy density at distant times.

Finally, in terms of practical applications our approach is currently limited to massless fields in even dimensions (see, however, the comments at the end of the introduction). Nonetheless the general viewpoint is very natural and could well form part of a deeper understanding of quantum inequalities and their ramifications.

APPENDIX A: PROOF OF LEMMA 3.1

We begin by considering the case $I = \mathbb{R}$. To simplify the notation, we write $A_{\mathbb{R}}^{(m)}$ as $A^{(m)}$ etc.

Lemma A.1. Suppose $\rho \in W^{-m,\infty}(\mathbb{R})$ (respectively, $\rho \in W^{-m,2}(\mathbb{R}) \cap W^{-m,\infty}(\mathbb{R}) + (W^{-m,\infty}(\mathbb{R}))_\epsilon$) is real. Then its associated quadratic form is $A^{(m)}$-form bounded (respectively, $A^{(m)}$-form compact).

Proof. Let ρ be an arbitrary element of $W^{-m,\infty}(\mathbb{R})$. We have

$$|\rho(f,f)| = |\langle |f|^2; \rho \rangle| \leq \|\rho\|_{-m,\infty} \| |f|^2 \|_{m,1} \tag{A1}$$

by definition of $\| \cdot \|_{-m,\infty}$. Now (by Theorem 4.13 in [25], Leibniz' rule, and the Cauchy-Schwarz inequality) there exists a $C > 0$ such that

$$\| |f|^2 \|_{m,1} \leq C \left(\|D^m f\|_2^2 + \|f\|_2^2 \right) = C\|f\|_{+1}^2, \qquad \forall f \in W^{m,2}(\mathbb{R}), \tag{A2}$$

so we have

$$|\rho(f,f)| \leq C\|\rho\|_{-m,\infty}\|f\|_{+1}^2. \tag{A3}$$

This has two consequences. First, ρ is $A^{(m)}$-form bounded, which proves the first part of the statement. Second, the self-adjoint operator R on $\mathcal{H}_{+1}$ associated with ρ by

$$\langle f|Rg \rangle_{+1} = \rho(f,g), \qquad \forall f,g \in \mathcal{H}_{+1} \tag{A4}$$

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has operator norm $\|R\| \leq C\|\rho\|_{-m,\infty}$. Accordingly, if $\rho_n \to \rho$ in $W^{-m,\infty}(\mathbb{R})$, their corresponding operators obey $R_n \to R$ in norm.

In particular, if $\rho \in W^{-m,2}(\mathbb{R}) \cap W^{-m,\infty}(\mathbb{R}) + (W^{-m,\infty}(\mathbb{R}))_\epsilon$, its corresponding operator may be regarded as a norm limit of a sequence of operators derived from distributions in $W^{-m,2}(\mathbb{R}) \cap W^{-m,\infty}(\mathbb{R})$. We will see presently that such operators are Hilbert-Schmidt; accordingly their norm limit is compact, thus proving that ρ is $A^{(m)}$-form compact.

To complete the proof, then, suppose $\rho \in W^{-m,2}(\mathbb{R}) \cap W^{-m,\infty}(\mathbb{R})$. In particular (cf. Theorem 7.63 in [25]) ρ is a tempered distribution whose Fourier transform $\widehat{\rho}$ is a measurable function obeying

$$\int \frac{|\widehat{\rho}(k)|^2}{1 + k^{2m}}\, dk < \infty. \tag{A5}$$

Let f, g be Schwartz test functions. We have

$$\langle g | Rf \rangle_{+1} = \langle g\overline{f}; \rho \rangle = \int \frac{dk}{2\pi}\, \overline{\widehat{g f}(k)}\widehat{\rho}(k) \tag{A6}$$

by Parseval's theorem, so

$$\langle g | Rf \rangle_{+1} = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi}\, \overline{\widehat{g}(k')}\widehat{\rho}(k)\widehat{f}(k' - k). \tag{A7}$$

We may interchange the order of integration by Fubini's theorem (which is justified owing to Eq. (A5) and the rapid decay of $\widehat{f}, \widehat{g}$) and, since the resulting expression holds for arbitrary g, identify

$$(1 + k'^{2m})\widehat{Rf}(k') = \int \frac{dk}{2\pi}\, \widehat{\rho}(k)\widehat{f}(k' - k) = \int \frac{dk}{2\pi}\, \widehat{\rho}(k' - k)\widehat{f}(k) \tag{A8}$$

for almost all k'.

It remains to observe that, by Fubini's theorem,

$$\int dk\, dk'\, \frac{|\widehat{\rho}(k - k')|^2}{(1 + k^{2m})(1 + k'^{2m})} = \int du\, \frac{|\widehat{\rho}(u)|^2}{1 + u^{2m}} \int dk\, \frac{1 + u^{2m}}{(1 + k^{2m})[1 + (u - k)^{2m}]} \tag{A9}$$

and that the k-integral defines a bounded function of u, to establish that R agrees with a Hilbert-Schmidt operator on Schwartz test functions. Since these are dense in $\mathcal{H}_{+1}$ we conclude that R is Hilbert-Schmidt, as required. $\square$

Proof of Lemma 3.1. Let $\mathcal{H}_{+1}$ and $\mathcal{H}_{+1,I}$ be the form domains $Q(A^{(m)}) = W^{m,2}(\mathbb{R})$ and $Q(A_I^{(m)}) = W_0^{m,2}(I)$ equipped with the appropriate form inner products. It is easy to show that

$$R_I = \jmath^* R \jmath \tag{A10}$$

where R and R_I are the self-adjoint operators on $\mathcal{H}_{+1}$ and $\mathcal{H}_{+1,I}$ associated with $\rho_{\mathbb{R}}$ and ρ_I respectively and $\jmath : \mathcal{H}_{+1,I} \to \mathcal{H}_{+1}$ is the isometry induced by the embedding of $W_0^{m,2}(I)$ in $W^{m,2}(\mathbb{R})$ described below Eq. (3.1).

It follows that R_I inherits any of the properties of boundedness, compactness, or positivity possessed by R. Thus Lemma 3.1 follows from Lemma A.1. $\square$

APPENDIX B: A CONVERGENCE RESULT

Theorem B.1. Suppose $\rho \in \mathcal{W}_m$ is the limit in the $\|\cdot\|_{-m,\infty}$-norm of a sequence $\rho_n \in \mathcal{W}_m$. Let $H^{(m)}$ and $H_n^{(m)}$ be the self-adjoint operators on $L^2(\mathbb{R})$ constructed by Lemma 3.1 and Theorem 2.4 applied to ρ and ρ_n respectively. Then $H_n^{(m)} \to H^{(m)}$ in the norm resolvent sense.

Proof. In the notation of Lemma A.1 and by the comments following Eq. (A4), the operators $R_n \to R$ in norm on $\mathcal{H}_{+1}$. Setting $F = (A^{(m)} + \mathbb{1})^{-1/2}$ this entails $FH_n^{(m)}F \to FH^{(m)}F$ in the operator norm on $L^2(\mathbb{R})$ and also implies that the $H_n^{(m)}$ and $H^{(m)}$ have a common lower bound, say $-M$. The result now follows (cf. Theorem VIII.25 in [27]) by the calculation

$$(H_n^{(m)} + t\mathbb{1})^{-1} = F \left[t\mathbb{1} + F \left(H_n^{(m)} - tA^{(m)} \right) F \right]^{-1} F$$
$$\longrightarrow F \left[t\mathbb{1} + F \left(H^{(m)} - tA^{(m)} \right) F \right]^{-1} F$$
$$= (H^{(m)} + t\mathbb{1})^{-1} \tag{B1}$$

for any $t > |M|$, with convergence in operator norm on $L^2(\mathbb{R})$. $\square$

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